Class Exercise 9

1. Find the area of the ellipse cut from the plane $z = cx$ by the cylinder $x^2 + y^2 = 1$.

   Solution. A parametrization is given by

$$\mathbf{s}(r,\theta) = (r\cos\theta, r\sin\theta, cr\cos\theta) \ , \quad (r,\theta) \in [0,1] \times [0,2\pi] \ .$$

   We have $\mathbf{s}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + c\cos\theta\mathbf{k}$ and $\mathbf{s}_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} - cr\sin\theta\mathbf{k}$. Hence

$$\mathbf{s}_r \times \mathbf{s}_\theta = -cr\mathbf{i} + r\mathbf{k} \ .$$

   The area of the ellipse is equal to

$$\iint_{D_1} |\mathbf{s}_r \times \mathbf{s}_\theta| \, dA(r,\theta) = \int_0^{2\pi} \int_0^1 \sqrt{1+c^2} \ r \, dr d\theta = \sqrt{1+c^2} \ \pi \ .$$

2. Find the flux of $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the plane $x = 0, x = 1$, and $z = 0$.

   Solution. A parametrization of the surface is

$$\mathbf{r}(x,y) = (x, y, 4 - y^2) \ , \quad (x,y) \in R \equiv [0,1] \times [-2,2] \ .$$

   As $\mathbf{r}_x \times \mathbf{r}_y = -\varphi_x\mathbf{i} - \varphi_y\mathbf{j} + \mathbf{k} = 2y\mathbf{j} + \mathbf{k}$, the flux is equal to

$$\begin{aligned}
& \iint_R \mathbf{F}(x,y,\varphi(x,y)) \cdot (2y\mathbf{j} + \mathbf{k}) \, dA(x,y) \\
= & \iint_R (2xy - 3(4-y^2)) \, dA(x,y) \\
= & \int_0^1 \int_{-2}^2 (2xy - 3(4-y^2)) \, dy dx \\
= & -32 \ .
\end{aligned}$$